04.05.25

The Gauß-Manin connection

Setup:

rel. norm. crossing divisor w.r.t S.

$$\begin{tikzcd} D \arrow[r, hook, "i"] & X \arrow[d, "f"] \\ & S \arrow[d, "g"] \\ & T \end{tikzcd}$$

smooth

Write

$$\Omega^\bullet_{X/S} = \Omega^\bullet_{X/S}(\log D)$$

$$\Omega^\bullet_{X/T} = \Omega^\bullet_{X/T}(\log D)$$

$\rightsquigarrow$ recall some motivation here from topology w. fiber bundles.

Goal: def'n $\nabla = \nabla^{GM}$ Gauß-Manin conn. on

$$H^n_{dR}(X/S) := R^n f_* \Omega^\bullet_{X/S} \in \mathrm{Qcoh}(S).$$

Constr. of ∇^{GM}

Have ex. seq.

See Hartshorne AG, Exc. II.5.16,

$$0 \to f^*\Omega^1_{S/T} \to \Omega^1_{X/T} \to \Omega^1_{X/S} \to 0.$$

$\Rightarrow \Omega^\bullet_{X/T}$ inherits Koszul filtration

$$\Omega^\bullet_{X/T} = K^0 \supset K^1 \supset K^2 \supset \dots,$$

where

$$K^{i,j} = \mathrm{im}\left(\Omega^{j-i}_{X/T} \otimes_{\mathcal{O}_X} f^*\Omega^i_{S/T} \xrightarrow{\wedge} \Omega^j_{X/T}\right).$$

Additionally,

$$gr^i := K^i/K^{i+1} \xrightarrow{\simeq} f^*\Omega^i_{S/T} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[-i]$$

"Spectral sequence of filtered object:" EGA III.1 (13.6.4)

$$E_1^{pq} = R^{p+q} f_* \, gr^p \Rightarrow R^{p+q} f_* \Omega^\bullet_{X/T}. \quad ⊛$$

We compute

$$E_1^{pq} = R^{p+q} f_* \, gr^p \simeq R^{p+q} f_*\left(f^*\Omega^p_{S/T} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[-p]\right)$$

$$\simeq R^q f_*\left(f^*\Omega^p_{S/T} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\right)$$

$$\simeq \Omega^p_{S/T} \otimes_{\mathcal{O}_S} R^q f_*(\Omega^\bullet_{X/S}) \quad \text{(proj. formula)}$$

So spec. seq. ⊛ becomes

$$E_1^{pq} = \Omega^p_{S/T} \otimes_{\mathcal{O}_S} R^q f_*(\Omega^\bullet_{X/S}) \Rightarrow R^{p+q} f_* \Omega^\bullet_{X/T}.$$

From E_1-page get complex

$$R^q f_*(\Omega^\bullet_{X/S}) \xrightarrow{d_1^{0,q}} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} R^q f_*(\Omega^\bullet_{X/S}) \xrightarrow{d_1^{1,q}} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} R^q f_*(\Omega^\bullet_{X/S}) \to \dots$$

Prop./Def. $d_1^{0,q}: R^q f_* \Omega^\bullet_{X/S} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} R^q f_* \Omega^\bullet_{X/S}$ is a flat connection, the Gauß-Manin connection.

Also write $d_1^{0,q} = \nabla^{GM}$.

Rmk. $d_1^{0,q}$ is also the boundary map in les obtained by applying Rf_* to

$$\begin{tikzcd} 0 \arrow[r] & gr^1 \arrow[r] \arrow[d, phantom, sloped, "\cong"] & K^0/K^2 \arrow[r] \arrow[d, equal] & gr^0 \arrow[r] \arrow[d, phantom, sloped, "\cong"] & 0 \\ 0 \arrow[r] & f^*\Omega^1_{S/T} \otimes \Omega^\bullet_{X/S}[-1] \arrow[r] & K^0/K^2 \arrow[r] & \Omega^\bullet_{X/S} \arrow[r] & 0 \end{tikzcd}$$

Proof. Note:

- $K^\bullet$ is mult.: $K^{i,l} \wedge K^{j,m} \subset K^{i+j,\, l+m}$

$$\rightsquigarrow gr^p \otimes_{f^{-1}\mathcal{O}_S} gr^{p'} \xrightarrow{-\wedge-} gr^{p+p'}$$

- $\cup$-product on $R^\bullet f_*$:

$$R^i f_* gr^p \otimes_{\mathcal{O}_S} R^j f_* gr^{p'} \xrightarrow{-\cup-} R^{i+j} f_*\left(gr^p \otimes_{f^{-1}\mathcal{O}_S} gr^{p'}\right) \to R^{i+j} f_* gr^{p+p'}.$$

It follows that (E_r^{pq}) has a mult. str.:

$$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',\, q+q'},$$

which satisfies e.g.

$$x \cdot y = (-1)^{(p+q)(p'+q')} y \cdot x; \quad (1)$$

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p+q} x \cdot d_r(y). \quad (2)$$

For $w \in \Omega^p_{S/T}$, can consider $w \otimes 1 \in \Omega^p_{S/T} \otimes H^0_{dR}(X/S)$

and then $d_1^{p,0}(w \otimes 1) = dw \otimes 1$.

From (2): for $f \in \mathcal{O}_S$, $\alpha \in H^q_{dR}(X/S)$:

$$\nabla^{GM}(f\alpha) = d_1^{0,q}(\underbrace{f \otimes 1}_{\in E_1^{0,0}} \cdot \underbrace{\alpha}_{\in E_1^{0,q}})$$

$$= d_1^{0,0}(f \otimes 1) \cdot \alpha + f \otimes 1 \cdot d_1^{0,q}(\alpha)$$

$$= df \otimes \alpha + f \nabla^{GM}(\alpha).$$

Also for $w \in \Omega^p_{S/T}$, $\alpha \in H^q_{dR}(X/S)$,

$$d_1^{p,q}(w \otimes \alpha) = d_1^{p,q}(w \otimes 1 \cdot \alpha)$$

$$= d_1^{p,0}(w \otimes 1) \cdot \alpha + (-1)^p (w \otimes 1) \cdot d_1^{0,q}(\alpha)$$

$$= dw \otimes \alpha + (-1)^p w \wedge \nabla^{GM}(\alpha),$$

and so

$$\nabla^{GM} \circ \nabla^{GM} = d_1^{1,q} \circ d_1^{0,q} = 0. \quad \boxtimes$$

Griffiths transversality.

$\Omega^\bullet_{X/S}$ also carries Hodge-filtration:

$$\Omega^\bullet_{X/S} \supset \dots \supset F^{i-1} \supset F^i \supset F^{i+1} \supset \dots,$$

where $F^i = \sigma_{\geq i} \Omega^\bullet_{X/S}$.

$\rightsquigarrow H^*_{dR}(X/S)$ inherits Hodge filtration.

Q.: "Is ∇^{GM} compatible w. $F^\bullet$? I.e., do we have

$$\nabla^{GM}_v(F^i) \subset F^i \text{?"}$$

would really like to say more here!

No: E.g. Picard-Fuchs equation.

Thm. (Griffiths transversality)

$$\nabla^{GM}(F^i) \subset F^{i-1} \otimes_{\mathcal{O}_S} \Omega^1_{S/T}.$$

Proof From Rmk we have ex. seq.

$$0 \to f^*\Omega^1_{S/T} \otimes \Omega^\bullet_{X/S}[-1] \to K^0/K^2 \to \Omega^\bullet_{X/S} \to 0. \quad ⊛$$

Observe: For $\mathcal{C}^\bullet \in ch(\mathcal{A})$ a complex:

$$\sigma_{\geq i}(\mathcal{C}^\bullet[-1]) = (\sigma_{\geq i-1} \mathcal{C}^\bullet)[-1].$$

Applying $\sigma_{\geq i}$ to ⊛ we get

$$\begin{tikzcd} 0 \arrow[r] & (f^*\Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^{i-1})[-1] \arrow[r] \arrow[d] & \sigma_{\geq i}(K^0/K^2) \arrow[r] \arrow[d] & F^i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f^*\Omega^1_{S/T} \otimes_{\mathcal{O}_S} \Omega^\bullet_{X/S}[-1] \arrow[r] & K^0/K^2 \arrow[r] & \Omega^\bullet_{X/S} \arrow[r] & 0 \end{tikzcd}$$

$\overset{Rf_*}{\rightsquigarrow}$

$$\begin{tikzcd} R^q f_* F^i \arrow[r, "\partial"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} R^q f_* F^{i-1} \arrow[d] \\ R^q f_* \Omega^\bullet_{X/S} \arrow[r, "\nabla^{GM}"] & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} R^q f_* \Omega^\bullet_{X/S}. \end{tikzcd}$$

$\boxtimes$

§. GM through Čech-cocycles

$X \xrightarrow{f} S$ smooth separated morph. of fin. type $\mathbb{C}$-schemes, S affine, $\mathcal{U} = \{U_i\}_{i\in I}$ affine (total ordering) open cover of X. Let (sheafy version!)

$$\mathcal{C}^{p,q}(\mathcal{U}) := \check{\mathcal{C}}^p(\mathcal{U}, \Omega^q_{X/S}) = \prod_{\alpha_0<\dots<\alpha_p} j_*\Omega^q_{X/S}|U_{\alpha_0\dots\alpha_p}.$$

"double Čech complex".

Recall $\Omega^\bullet_{X/S} \xrightarrow[\sim]{q.\,iso} \mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}(\mathcal{U}))$

resolution by Rf_*-acyclic sheaves

$$\Rightarrow R^if_*(\Omega^\bullet_{X/S}) \xrightarrow{\simeq} H^i\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}(\mathcal{U}))$$

Let $v \in \Omega^{1,\vee}_{S/\mathbb{C}}(S)$ v. field, and spse we have lifts $\tilde{v}$ $\{v_i \in \Omega^{1,\vee}_{X/\mathbb{C}}(U_i)\}$ (maybe after shrinking U_i's)

on U_i: "Lie deriv." → map of complexes

v_i lifts v on S $\Rightarrow \mathcal{L}_{v_i}: \Omega^q_{X/S}|U_i \to \Omega^q_{X/S}|U_i$

$\varphi\, dx_1\wedge\dots\wedge dx_q \mapsto v_i(\varphi)\, dx_1\wedge\dots\wedge dx_q$ — $x_1,\dots,x_n$ étale coord w.r.t. S on U_i s.t. $dx_i(v_i)=0$.

on $U_i - U_j$: "contraction"

$v_i - v_j$ is vertical $\Rightarrow (v_i-v_j)\lrcorner : \Omega^q_{X/S}|U_{ij} \to \Omega^{q-1}_{X/S}|U_{ij}$

$\omega \mapsto \omega(v_i-v_j,\dots)$.

- Define $\mathcal{L}_{\tilde v}: \mathcal{C}^{\bullet,\bullet}(\mathcal{U},\Omega^\bullet_{X/S}) \to \mathcal{C}^{\bullet,\bullet}(\mathcal{U},\Omega^\bullet_{X/S})$ by $\Omega^q_{X/S}|U_{\alpha_0<\dots<\alpha_p} \xrightarrow{\mathcal{L}_{v_{\alpha_0}}} \Omega^q_{X/S}|U_{\alpha_0<\dots<\alpha_p}$.
- Define

$$H_{\tilde v}: \mathcal{C}^{\bullet,\bullet}(\mathcal{U},\Omega^\bullet_{X/S}) \to \mathcal{C}^{\bullet+1,\bullet-1}(\mathcal{U},\Omega^\bullet_{X/S})$$

by $H_{\tilde v}(\sigma)_{\alpha_0<\dots<\alpha_{p+1}} = (-1)^q(v_{\alpha_0}-v_{\alpha_1})\lrcorner\,\sigma_{\alpha_1<\dots<\alpha_{p+1}}$,

where $\sigma \in \mathcal{C}^{p,q}$, $(\sigma_{\alpha_1<\dots<\alpha_{p+1}} \in \Omega^q_{X/S}|U_{\alpha_1\dots\alpha_{p+1}})$

Now get a map of complexes

$$H_{\tilde v} + \mathcal{L}_{\tilde v}: \mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}) \to \mathrm{Tot}(\mathcal{C}^{\bullet,\bullet})$$

Thm (Katz-Oda)

$$\begin{tikzcd} R^if_*\Omega^\bullet_{X/S} \arrow[r, "\nabla^{GM}_v"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\oslash"] & R^if_*\Omega^\bullet_{X/S} \arrow[d, "\simeq"] \\ H^i\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}) \arrow[r, "H^i(H_{\tilde v}+\mathcal{L}_{\tilde v})"'] & H^i\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}). \end{tikzcd}$$

$\boxtimes$

Remark

- $\mathcal{L}_{v_i} - \mathcal{L}_{v_j} = d\circ(v_i-v_j)\lrcorner + (v_i-v_j)\lrcorner\circ d$; (Cartan's magic formula)
- $(v_i-v_j)\lrcorner + (v_j-v_k)\lrcorner$ (cocycle condition)

⇝ much more general mechanism at play to obtain map of total complexes to data like this

§ Regularity Theorem

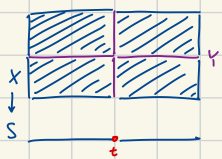
Thm Let $f: X \to S$ smooth morph. of fin. type smooth $\mathbb{C}$-schemes, X = complement of rel. normal crossing divisor in smooth proper S-scheme. Then $(R^if_*\Omega^\bullet_{X/S}, \nabla^{GM})$ has regular singularities.

Proof Sketch

Reduce to following case

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & \bar X \arrow[d, "\bar f"] & Y \arrow[l, hook'] & \bar f^{-1}(T) \arrow[l, hook'] \arrow[d] \\ S \arrow[r, hook] & \bar S & & T \arrow[ll, hook'] \end{tikzcd}$$

Here S curve, $\bar S \supset S$ compactification, $T = \bar S \setminus S$, $X \hookrightarrow \bar X$ compactification, $Y = \bar X \setminus X$ normal crossing, $\bar X \xrightarrow{\bar f} \bar S$ smooth proper, $Y \cap \bar f^{-1}(s)$ rel. normal crossing.



Consider $R^i\bar f_*\Omega^\bullet_{\bar X/\bar S}(\log Y)$ coh. on $\bar S$.

Fact $R^i\bar f_*\Omega^\bullet_{\bar X/\bar S}(\log Y)|S \overset{nat.}{\simeq} R^if_*\Omega^\bullet_{X/S}$.

Set $R^i_0(\bar f) = R^i\bar f_*\Omega^\bullet_{\bar X/\bar S}(\log Y)/\text{torsion}$.

loc. free $+\ R^i_0(\bar f)|S \simeq R^if_*\Omega^\bullet_{X/S}$.

Now let $t \in T$, v on $\bar S$ v. field on nghb. of t s.t. v vanishes at t. (i.e., $v \in \mathrm{Der}_t(\bar S/\mathbb{C})$)

Goal: "Extend $\nabla^{GM}_{v|S}: R^if_*\Omega^\bullet_{X/S} \to R^if_*\Omega^\bullet_{X/S}$ to $\nabla_v: R^i_0(\bar f) \to R^i_0(\bar f)$."

Let $\{U_i\}$ open affine covering of $\bar X$ and $\forall i$, let v_i v. field on U_i lifting v s.t.

$$\langle v_i, \Omega^1_{\bar X/\mathbb{C}}(\log Y)\rangle \subset \mathcal{O}_{U_i}. \quad ⊛$$

⇝ $v_i\lrcorner\,\Omega^q_{\bar X/\mathbb{C}}(\log Y) \subset \Omega^{q-1}_{\bar X/\mathbb{C}}(\log Y)$

and so

$$\mathcal{L}_{v_i} = d\circ(v_i\lrcorner) + (v_i\lrcorner)\circ d: \Omega^q_{\bar X/\mathbb{C}}(\log Y) \to \Omega^q_{\bar X/\mathbb{C}}(\log Y)$$

is defined. We have

$\mathcal{L}_{v_i}: \Omega^\bullet_{\bar X/\bar S}(\log Y) \to \Omega^\bullet_{\bar X/\bar S}(\log Y)$;

$(v_i-v_j)\lrcorner: \Omega^\bullet_{\bar X/\bar S}(\log Y) \to \Omega^\bullet_{\bar X/\bar S}(\log Y)$.

By similar mechanism to before: this datum induces $R^i_0(\bar f) \to R^i_0(\bar f)$ extending ∇^{GM}_v by Katz-Oda. $\boxtimes$

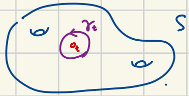
⊛ works because of s.e.s

$$0 \to \bar f^*\Omega^1_{\bar S/\mathbb{C}}(\log t) \to \Omega^1_{\bar X/\mathbb{C}}(\log Y) \to \Omega^1_{\bar X/\bar S}(\log Y) \to 0$$

arbitrary v_i does not work: e.g. $\partial_z \lrcorner \frac{1}{z}dz\wedge dw = \frac{1}{z}dw$ would not be a form in $\Omega^1_{\mathbb{A}^2/\mathbb{C}}(\log\{z=0\})$.

§ Cor: Quasi-unipotent monodromy at ∞.

$S/\mathbb{C}$ smooth curve, $S \hookrightarrow \bar{S}$ comp., $\mathbb{L}$ local system on $S(\mathbb{C})$, $t \in T = \bar{S} \setminus S$ a point, $\gamma_t \in \pi_1(S(\mathbb{C}), x)$ "monodromy of t":



Get $\overset{M_t}{\overset{\shortparallel}{\rho_{\mathbb{L}}(\gamma_t)}} \in GL(\mathbb{L}_x)$.

Def. $\mathbb{L}$ is quasi-unipotent at t if $\exists\, e \geq 1$ s.t. M_t^e is unipotent

$\Leftrightarrow$ all $\lambda \in \mathbb{C}^\times$ eigenvalues of M_t are roots of unity.

Thm. (Brieskorn) $f: X \to S$ a smooth morphism. Suppose $R^i f_* \mathbb{C}$ a local system (e.g. f proper, or X compl. of rel. normal crossing divisor). Then $R^i f_* \mathbb{C}$ is q.-unipotent at ∞.

Proof ultimately comes down to:

- regularity of ∇^{GM};
- #-theoretical fact: "$\alpha \in \mathbb{C}$ s.t. α and $\exp(2\pi i \alpha)$ algebraic $\Rightarrow \alpha \in \mathbb{Q}$."

§ Motivation for ∇^{GM}

In topology: let $\pi: E \to B$ fiber bundle of "nice spaces". Then $R^i\pi_* \mathbb{C}_E$ local system on B w. $(R^i\pi_* \mathbb{C}_E)_x = H^i_{sing.}(\pi^{-1}(x); \mathbb{C})$.

In alg. geom.: $\pi: X \to S$ proper smooth morph. of $\mathbb{C}$-var's. Ehresmann $\Rightarrow \pi$ is a fiber bundle $\Rightarrow R^i\pi_*\mathbb{C}_X$ local system.

Q.: "Corresponding flat ∇?"

Note: $H^i_{dR}(X/S) = R^i\pi_*\Omega^\bullet_{X/S}$ has

$H^i_{dR}(X/S)_x = H^i_{dR}(\pi^{-1}(x); \mathbb{C}) \simeq H^i_{sing}(\pi^{-1}(x); \mathbb{C})$.

Goal: "define ∇ on $H^i_{dR}(X/S)$."

§ Picard-Fuchs equation

$\mathcal{E} \xrightarrow{\pi} S = \mathbb{A}^1 - \{0,1\} = \operatorname{Spec} \overset{R}{\overset{\shortparallel}{\mathbb{C}[\lambda, \lambda^{-1}(\lambda-1)^{-1}]}}$,

$\mathcal{E} = \operatorname{Proj} R[x,y,z] / (y^2 z = x(x-z)(x-\lambda z))$.

Have $R^0\pi_*\Omega^1_{\mathcal{E}/S} = R^1\pi_*\Omega^1_{\mathcal{E}/S}[-1]$

$= R^1\pi_*\sigma_{\geq 1}\Omega^\bullet_{\mathcal{E}/S} \to R^1\pi_*\Omega^\bullet_{\mathcal{E}/S}$.

Let $\omega = \frac{dx}{y} \in \pi_*\Omega^1_{\mathcal{E}/S}(S)$, $\frac{d}{d\lambda} \in \Omega^{1,\vee}_{S/\mathbb{C}}(S)$.

Set $\omega^{(1)} = \frac{d}{d\lambda}\omega$ to be im. of

$$R^0\pi_*\Omega^1_{\mathcal{E}/S} \to R^1\pi_*\Omega^\bullet_{\mathcal{E}/S} \xrightarrow{\nabla^{GM}} \Omega^1_{S/\mathbb{C}} \otimes_{\mathcal{O}_S} R^1\pi_*\Omega^\bullet_{\mathcal{E}/S} \xrightarrow{\langle \frac{d}{d\lambda}, - \rangle} R^1\pi_*\Omega^\bullet_{\mathcal{E}/S}.$$

Then ω and $\omega^{(1)}$ generate $R^1\pi_*\Omega^\bullet_{\mathcal{E}/S}$ as an $\mathcal{O}_S$-module.